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Unruh effect for circular motion in a cavity

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Abstract. A quantum detector moves in a circle of radius r, within a concentric circular cavity of radius R, with constant angular velocity Ω. A quantum field, enclosed in that two-dimensional cavity, is in its ground state. We compute the probability that the detector will be excited by the vacuum fluctuations of the field. If $R < c/\Omega$, the detector is not excited. For larger R the excitation rate is a chaotic function of R and a multiply periodic function of the observation time t. However, for $R \gg ct \gg r$ the excitation rate tends to a finite limit, independent of R and t.

1. Introduction

It is well known that a quantum detector moving with a constant acceleration in a Minkowski vacuum reacts as if it were in a thermal bath [1, 2]. This is called the Unruh effect. It results from the autocorrelation of the field variables along the world line of the detector. The vacuum fluctuations appear to have a Planckian spectrum, with a temperature $kT = g\hbar/2\pi c$.

For any reasonable linear acceleration this temperature is exceedingly low. Yet Bell *et al* [3–5] attempted to verify this effect and performed extensive calculations on the quantum fluctuations of electron orbits in high-energy storage rings, where the centripetal acceleration is much higher than the one obtainable in a linear accelerator. They found that circular motion led to a similar effect, and discussed possible observable consequences. More recently, Rogers [6] proposed observing this effect by using a single electron revolving in a Penning trap. Other authors [7, 8] investigated the behaviour of rotating quantum detectors by using rotating coordinate systems, with contradictory results.

None of these authors took into consideration the fact that the properties of a quantum field in a closed cavity (such as a storage ring or a Penning trap) are radically different from those of a free field in unbounded space. The presence of boundaries affects the dynamical properties of a quantum field by altering the frequencies of its normal modes. Finite size effects have been known for a long time, both theoretically [9] and experimentally [10]. In this paper we show that the effects of a finite cavity size cannot be ignored, unless that size is much larger than ct, where t is the duration of the circular motion. This condition is not satisfied in the experiments discussed by Bell *et al*, and by Rogers.

We investigate this issue by means of a simple theoretical model involving a massless scalar field ϕ enclosed in a two-dimensional circular cavity of radius R. In spite of these radical simplifications (no third space dimension and no polarization) the calculations are fairly complex, because they involve six different parameters with the dimension of length: the orbit radius r, the cavity radius R, the wavelength for transitions between levels of the detector, the Compton wavelength of the detector, and (apart from a factor c) the observation time t and the period of revolution $2\pi/\Omega$. This gives five independent dimensionless ratios.

With reasonable values for the physical parameters, some of these dimensionless ratios are very large, and among them some may be very much larger than others. These widely different orders of magnitude will allow us to make various approximations and to obtain a closed expression for the transition rate.

Section 2 of this paper discusses properties of a two-dimensional scalar field enclosed in a circular cavity, and ends with an explicit expression for the Wightman function. Section 3 is a general treatment of transitions between discrete levels, induced by a periodic perturbation. Section 4 specifically deals with the detection of vacuum fluctuations. It is shown that the transition rate strongly depends on the cavity radius R, unless $R \gg ct$.

2. Scalar field in a circular cavity

We take as the free field Lagrangian density $\mathcal{L}_\phi = \frac{1}{2}[c^2\dot{\phi}^2 - (\nabla\phi)^2]$, which gives the equal time commutation relation

$$[\phi(\boldsymbol{r},t), \dot{\phi}(\boldsymbol{r}',t)] = i\hbar c^2\,\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{1}$$

and the field equation $\ddot{\phi} = c^2\nabla^2\phi$. We shall assume that $\phi = 0$ at the cavity's boundary, $|\boldsymbol{r}| = R$. There is an apparent contradiction between this condition and equation (1). The difficulty originates in the expansion of ϕ into normal modes

$$\phi(\boldsymbol{r},t) = \sum_\omega \mathrm{e}^{-\mathrm{i}\omega t} f_\omega(\boldsymbol{r})\, a_\omega + \mathrm{h.c.} \tag{2}$$

where each mode $f_\omega(\boldsymbol{r})$ satisfies $\nabla^2\phi + (\omega/c)^2\phi = 0$. In polar coordinates, r, θ, these modes are $f_\omega(\boldsymbol{r}) \sim \mathrm{e}^{\mathrm{i}m\theta} J_m(\omega r/c)$, and the boundary condition $\phi(R,\theta) = 0$ is satisfied by taking

$$\omega = \frac{c\, z_{ms}}{R} \tag{3}$$

where z_{ms} is the sth zero of the Bessel function $J_{|m|}(z)$. The eigenfunctions $f_\omega(\boldsymbol{r})$ are mutually orthogonal and may be normalized by $\int f_\omega^*(\boldsymbol{r})\, f_{\omega'}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} = \delta_{\omega\omega'}$. Since they form a complete set, they also satisfy

$$\sum_\omega f_\omega^*(\boldsymbol{r})\, f_\omega(\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{4}$$

It is clear from equation (4) that a sum over modes can converge only in a distribution sense. This is also true for the sum in equation (2), and this explains the contradiction between equation (1) and the boundary condition $\phi(R,\theta) = 0$. One way of overcoming this difficulty is to impose an arbitrary high frequency cutoff on the sum over modes. We would then have, instead of $\delta(\boldsymbol{r}-\boldsymbol{r}')$, a rapidly oscillating function of r and $\boldsymbol{r}'$, with a sharp peak at $\boldsymbol{r} = \boldsymbol{r}'$, and the relativistic invariance would be impaired [11]. This is quite admissible, since the cavity's rest frame is a privileged frame in our problem. A fully relativistic treatment would thus have to describe the cavity itself as a dynamical system: instead of a formal boundary condition there would be a strongly repulsive interaction between the cavity walls and the field. We shall henceforth ignore these fine points—which are inherent in quantum field theory—because they are unlikely to affect the main results of our calculations.

Explicitly, the normalized modes are

$$f_{ms}(r,\theta) = \mathrm{e}^{\mathrm{i}m\theta}\,\frac{J_m(z_{ms}r/R)}{\sqrt{\pi}\,R\,J'_m(z_{ms})} \tag{5}$$

where a double index ms replaces the original index ω. The creation and annihilation operators of mode ms are normalized by

$$[a_{ms}, a^\dagger_{m's'}] = \delta_{mm'}\,\delta_{ss'}\,\frac{\hbar c^2}{2\omega_{ms}} \tag{6}$$

so that the commutation relation (1) is satisfied.

We shall later need explicitly the vacuum expectation value (Wightman function) $\langle 0|\phi(\boldsymbol{r}',t')\,\phi(\boldsymbol{r},t)|0\rangle$. It can be obtained from the preceding equations

$$\langle 0|\phi(\boldsymbol{r}',t')\,\phi(\boldsymbol{r},t)|0\rangle = \hbar\sum_\omega \mathrm{e}^{\mathrm{i}\omega(t-t')}\,\frac{f_\omega(\boldsymbol{r}')\,f_\omega(\boldsymbol{r})}{2\omega}\,. \tag{7}$$

In the special case of our circular cavity this becomes

$$\langle 0|\phi(r'\theta't)\,\phi(r\theta t)|0\rangle = \frac{\hbar c}{2\pi R}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m(\theta'-\theta)}\sum_s C_{ms}\,\mathrm{e}^{\mathrm{i}(t-t')cz_{ms}/R} \tag{8}$$

where

$$C_{ms} = \frac{J_m(z_{ms}r'/R)\,J_m(z_{ms}r/R)}{z_{ms}\,[J'_m(z_{ms})]^2}\,. \tag{9}$$

3. Transitions between discrete levels

We now turn our attention to the detecting process. The detector has discrete energy levels, and its interaction with the quantized field ϕ causes transitions between them. To determine the transition rate one cannot use the familiar Fermi golden rule, since the latter is valid only when there is a *continuum* of final states, and is not applicable to transitions between *discrete* levels. We therefore proceed as in our previous paper (Levin *et al* [12]) and repeat here only the main points of the argument given there, discussing only changes that are necessary for the present problem.

Consider in general two weakly coupled quantum systems, such as an 'atom' a (our detector) and a 'background' b, which may be a quantized field or any other agent whose interaction with the atom causes quantum transitions. In the absence of coupling the Hamiltonian is $H_0 = H_a + H_b$. We assume that H_a is time-independent and has a discrete spectrum: $H_a\,|m\rangle = E_m\,|m\rangle$. No such assumptions are made for H_b which may explicitly depend on time. The states of the background are described by an *arbitrary* orthonormal basis $|\alpha\rangle$, and those of the combined system by the tensor product of these two bases

$$|m\alpha\rangle \equiv |m\rangle \otimes |\alpha\rangle. \tag{10}$$

The coupling between the two systems is assumed to be a direct product $A \otimes B$, where the operators A and B belong to the atom and the background, respectively. The Schrödinger equation is

$$i\hbar \frac{d\psi}{dt} = (H_a + H_b + A \otimes B)\,\psi \tag{11}$$

where ψ is a linear combination of the basis vectors (10). The $\otimes$ sign will henceforth be omitted. The operators A and B were naturally written in equation (11) in their Schrödinger representation. We shall later also need the Heisenberg representation of B, which is

$$B_{\mathrm{H}}(t) = U^{\dagger}(t)\, B(t)\, U(t) \tag{12}$$

where $U(t)$ is a unitary operator satisfying $i\hbar\, dU(t)/dt = H_b\, U(t)$, with the initial condition $U(0) = 1$.

We now expand

$$\psi(t) = U(t) \sum_{m\alpha} e^{-iE_m t/\hbar}\, c_{m\alpha}(t)\, |m\alpha\rangle \tag{13}$$

where

$$c_{m\alpha}(t) = e^{iE_m t/\hbar}\, \langle m\alpha|U^{\dagger}(t)|\psi(t)\rangle. \tag{14}$$

The Schrödinger equation (11) is equivalent to

$$i\hbar \frac{dc_{m\alpha}(t)}{dt} = \sum_{n\beta} e^{i(E_m - E_n)t/\hbar}\, A_{mn}\, \langle\alpha|B_{\mathrm{H}}(t)|\beta\rangle\, c_{n\beta} \tag{15}$$

where $A_{mn} = \langle m|A|n\rangle$. The initial state of the combined system is given by equation (13) with $c_{i0}(0) = 1$, and all other $c_{m\alpha}(0) = 0$. Here the index 0 refers to the initial state of the background subsystem. That state will be denoted by $|0\rangle$. It may be the vacuum or any other state resulting from the physical preparation of that background. We are interested in the probability of finding the atom in a prescribed final state $|f\rangle$, at time t, irrespective of the final state of the background. That probability is $P(t) = \sum_\alpha |c_{f\alpha}(t)|^2$, and the transition rate is

$$\Gamma(t) = \frac{dP(t)}{dt} = \sum_{\alpha} c^{*}_{f\alpha}(t)\, \frac{dc_{f\alpha}(t)}{dt} + \text{c.c.}. \tag{16}$$

For the given initial conditions, and for any $|f\rangle$ orthogonal to the initial state of the atom, equation (15) becomes, in first order perturbation theory,

$$i\hbar \frac{dc_{f\alpha}(t)}{dt} = e^{i\omega t}\, A_{fi}\, \langle\alpha|B_{\mathrm{H}}(t)|0\rangle \tag{17}$$

where $\omega = (E_f - E_i)/\hbar$. It follows that

$$i\hbar\, c_{f\alpha}(t) = A_{fi} \int_{t_0}^{t} e^{i\omega t'}\, \langle\alpha|B_{\mathrm{H}}(t')|0\rangle\, dt' \tag{18}$$

and therefore

$$\Gamma(t) = |A_{fi}/\hbar|^2 \int_0^t \mathrm{e}^{\mathrm{i}\omega(t-t')} \sum_\alpha \langle 0|B_{\mathrm{H}}(t')|\alpha\rangle \langle\alpha|B_{\mathrm{H}}(t)|0\rangle \,\mathrm{d}t' + \text{c.c.}$$
$$= |A_{fi}/\hbar|^2 \int_0^t \mathrm{e}^{i\omega(t-t')} \langle 0|B_{\mathrm{H}}(t')B_{\mathrm{H}}(t)|0\rangle \,\mathrm{d}t' + \text{c.c.}. \quad (19)$$

This result is valid as long as $\sum_f P(t) \ll 1$. For longer times, first order perturbation theory becomes inadequate.

In a stationary situation, such as the one that we are considering,

$$W(t', t) = \langle 0|B_{\mathrm{H}}(t')B_{\mathrm{H}}(t)|0\rangle \quad (20)$$

depends only on the time difference $t - t'$ and can be written as

$$W(t', t) = W(t - t') = W^*(t' - t). \quad (21)$$

Introducing a new integration variable $\tau = t - t'$, we finally obtain

$$\Gamma(t) = |A_{fi}/\hbar|^2 \int_0^t \mathrm{e}^{\mathrm{i}\omega\tau}\, W(\tau)\,\mathrm{d}\tau + \text{c.c.}$$
$$= |A_{fi}/\hbar|^2 \int_{-t}^t \mathrm{e}^{\mathrm{i}\omega(t-t')}\,\mathrm{d}\tau. \quad (22)$$

4. Quantum detector in circular motion

We now introduce the detector, which we take as a point particle having two internal states (its internal structure will be described with notations appropriate to a spin-$\frac{1}{2}$ particle). A fully relativistic treatment of the detector interacting with the scalar field ϕ would necessitate describing it, too, by a quantized field. However, as shown in our previous paper [12], if the detector is massive enough to remain localised in a wave packet which is very small compared to the size of the cavity, and if we can neglect the creation of virtual pairs of detectors and antidetectors, it is an excellent approximation to treat classically the position of the detector, and to assume that it follows a *prescribed* trajectory $\boldsymbol{r} = \boldsymbol{r}(t)$.

In that case the only dynamical variables in the Hamiltonian are the quantized scalar field ϕ and the internal degrees of freedom of the detector. In order to use the results of the preceding section, we have to write the Hamiltonian in the Schrödinger representation. In the simplified model that we are considering, it is

$$H = H_\Phi + \omega\, S_z + \lambda\, S_x\, \Phi(\boldsymbol{r}, t) \quad (23)$$

where Φ is the Schrödinger representation of the scalar field (the symbol ϕ that was used in section 2 was its Heisenberg representation, namely $\phi = \Phi_{\mathrm{H}}$); the detector's internal variables S_x and S_z are the usual spin matrices (eigenvalues $\pm\hbar/2$); ω is a constant, such that $\hbar\omega$ is the energy separation of the two levels of the detector, and λ is the detector's coupling constant to the scalar field. This coupling causes transitions between the internal levels of the detector. The occurrence of a transition may be interpreted as the detection of a vacuum fluctuation of the field.

We can now use the results of the preceding section to evaluate the transition rate, $\Gamma(t)$, between the two levels of the detector. The symbols H_a and H_b that were used in that section correspond to ωS_z and H_Φ, respectively. As a simple example, we consider uniform circular motion: $r = \text{const}$, $\theta = \Omega t$.

In equation (19), we set

$$|A_{fi}|^2 = (\lambda S_x)^2 = \tfrac{1}{4}\hbar^2\lambda^2 \tag{24}$$

and we take for $W(t', t)$ the Wightman function given in equation (8), evaluated at a pair of points on the detector's trajectory. Explicitly, it is

$$\langle 0|\phi(r\theta' t')\,\phi(r\theta t)|0\rangle = \frac{\hbar c}{2\pi R}\sum_{ms} C_{ms}\,\mathrm{e}^{\mathrm{i}(t'-t)(m\Omega - cz_{ms}/R)} \tag{25}$$

where C_{ms} is defined by equation (9), with $r' = r$. Combining equations (20), (22) and (25), we obtain

$$\Gamma(t) = \frac{\lambda^2\hbar c}{8\pi R}\sum_{ms} C_{ms}\int_{-t}^{t} \mathrm{e}^{i\tau(\omega - m\Omega + cz_{ms}/R)}\,\mathrm{d}\tau. \tag{26}$$

If we now take the limit $t \to \infty$ as is customary, the above integral becomes $2\pi\delta(\omega - m\Omega + cz_{ms}/R)$, and it is obvious that the only values of m that are involved satisfy $m > \omega/\Omega$. For positive m, all the zeros of $J_m(z)$ satisfy $z_{ms} > m$ [13]. It follows that the argument of the delta function cannot vanish if $\Omega < c/R$, and therefore in that case $\Gamma = 0$.

The absence of transitions if $\Omega < c/R$ could have been foreseen as follows: A transformation to a rotating coordinate system ($\theta_{\text{rot}} = \theta - \Omega t$) is everywhere regular up to a radius c/Ω (at that radius, $g_{tt} = c^2 - \Omega^2 r^2$ changes sign). Thus, by using a rotating coordinate system in a cavity whose radius is smaller than c/Ω, the detector appears to be static, and the spacetime metric itself is stationary. Since this transformation does not mix the modes of the field, a vacuum with no mode excited remains a vacuum, and therefore the static detector cannot be excited. The same conclusion is obviously valid for any axially symmetric three-dimensional cavity, whose largest radius is less than c/Ω.

For $R > c/\Omega$ the situation becomes more complicated. If we interpret the delta function literally, Γ is strictly zero for nearly all values of the parameters, and it is infinite at some isolated points. This manifestly contradicts the conditions for validity of perturbation theory. We must therefore consider the situation for finite values of t. We then have

$$\Gamma(t) = \frac{\lambda^2\hbar c}{4\pi R}\sum_{ms} C_{ms}\,\frac{\sin t\alpha}{\alpha} \tag{27}$$

where

$$\alpha = \omega - m\Omega + \frac{cz_{ms}}{R}\,. \tag{28}$$

The sum (27) is illustrated in figure 1, where grey stripes indicate the regions in which $|\sin t\alpha/t\alpha| > 0.1$. Further stripes, corresponding to larger values of the denominator α, yield gradually decreasing contributions to the sum (27). For consecutive stripes $\Delta m \simeq \pi/\Omega t$, which is a very small number if the detector performs many revolutions

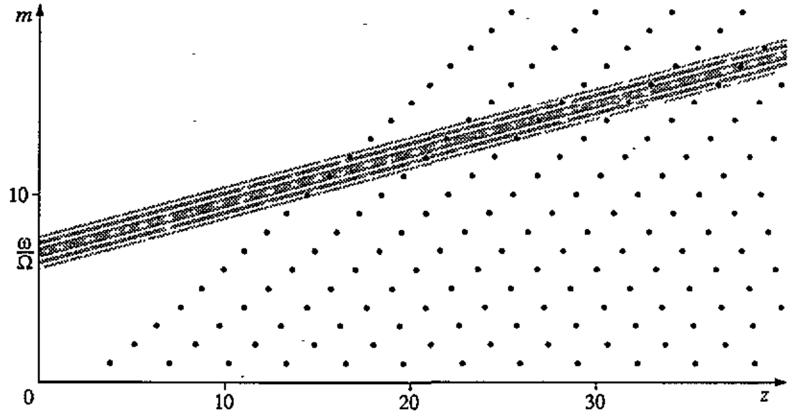

Figure 1. The sum in equation (27) involves all the points of the lattice where $z = z_{ms}$, but only those lying on the grey stripes give a substantial contribution. For reasonable physical parameters, m is very large, the stripes are considerably narrower than sketched here, and their actual slope, $c/\Omega R$, may be much smaller than in this figure. The values of z_{ms} actually used for this figure were computed by means of the approximate formula (A3) for $m = 1$ to 20. The errors, with respect to the known exact values of z_{ms}, are smaller than the size of the dots.

during the observation time t. The actual stripes are therefore much narrower than those sketched in the figure. Their slope is $c/\Omega R$, and their horizontal extent is $\Delta z = \pi R/ct$. On the other hand, the distance between consecutive values of z_{ms} (for given m) is always more than π (see appendix). Therefore the phases of consecutive terms in the sum (27) differ by $t\,\Delta\alpha \gtrsim \pi ct/R$, and different regimes must be distinguished, depending on whether the ratio R/ct is large or small (for intermediate values there is a continuous transition between the two extreme behaviours which cannot be described in any simple way).

If $R \lesssim ct$, the various $\sin t\alpha$ act incoherently and the sum (27) is a multiply periodic function of time. However, there are exceptional values of R for which one of the z_{ms} yields a very small value of α and the corresponding C_{ms}/α dominates the entire sum. The contribution of such a term is a periodic function of t, with a period $2\pi\alpha^{-1}$ and an amplitude $\sim\alpha^{-1}$. Both are rapidly varying functions of R. As seen in figure 1, there is an infinite number of these exceptional points for which α is very small. However, the coefficient C_{ms} itself is large only in a narrow domain of values of m (as explained below). Therefore, the values of R for which $\Gamma(t)$ is large are sparse and 'randomly' distributed. (If we had a three-dimensional cavity with height comparable to the radius R the result would be qualitatively similar, because the field would still have a discrete spectrum, with level separations of the same order of magnitude. However, for a very long cylindrical cavity, with height much larger than ct, the results are likely to be smooth functions, as in the case $R \gg ct$ which is discussed below.)

If $R \gg ct$, there are for each m many consecutive roots z_{ms} swept by the grey stripes in figure 1. The phases of consecutive terms of the sum differ by $\sim\pi ct/R \ll 1$, so that these terms act coherently. Moreover, the coefficients C_{ms}, which are given by equation (9), depend on z_{ms} in a very smooth way, as can be seen from the formulae given in the appendix. Therefore the sum (27) contains only contributions from terms in which $t\alpha$ is not a large number, because of the mutually cancelling oscillations of $\sin t\alpha$ that occur for larger values of α.

We then have

$$J_m(z_{ms}\, r/R) = J_m[(m\Omega - \omega + \alpha)\, r/c] \simeq J_m[(m\Omega - \omega)r/c]. \tag{29}$$

The error in the argument of J_m on the right-hand side of this equation is $\alpha r/c = (t\alpha)\,(r/ct)$. This is much smaller than the distance of consecutive zeros, $\Delta z_{ms} \gtrsim \pi$, because $t\alpha$ cannot be large, as explained above, while r/ct is a small number if the duration of the process extends over many periods of revolution. We can therefore write C_{ms}, which is defined by equation (9), as

$$C_{ms} = \{J_m[(m\Omega - \omega)r/c]\}^2\, \frac{\Delta z_{ms}}{2} \tag{30}$$

where use was made of equation (A7) in the appendix. The J_m^2 term in C_{ms} no longer depends on the index s and we can now perform explicitly the sum over s in (27)

$$\sum_s \Delta z_{ms}\, \frac{\sin t\alpha}{\alpha} \simeq \int_{-\infty}^{\infty} \mathrm{d}z\, \frac{\sin t\alpha}{\alpha} = \frac{R}{c} \tag{31}$$

where the limits of integration have been extended to $\pm\infty$ because of the sharp peak in the integrand. Note that the approximation in equation (31) is justified only if $\sin t\alpha$ changes slowly for consecutive s, which indeed is true if $R \gg ct$, but not if $R \lesssim ct$.

Let us introduce the notations $\beta = \Omega r/c < 1$ (this is the dimensionless velocity of the detector) and $\mu = \omega/\Omega$ (this is a very large number, if ω is a typical atomic frequency while Ω cannot exceed c/r). We have

$$J_m[(m\Omega - \omega)r/c] = J_m[m\beta\,(1 - \mu/m)]. \tag{32}$$

Recall that $m > \mu$, as can be seen from equation (29). We further define a variable ξ by

$$\operatorname{sech}\xi = \beta\,(1 - \mu/m) \tag{33}$$

and use the asymptotic formula [13]

$$[J_m(m\operatorname{sech}\xi)]^2 \simeq \frac{\mathrm{e}^{-2m(\xi - \tanh\xi)}}{2\pi m \tanh\xi}\,. \tag{34}$$

It will now be shown that this expression is sharply peaked in a narrow domain of values of m. This will allow us to expand the exponent on the right-hand side of equation (34) into a power series around its maximum, and to keep only two terms of this series.

From equation (33), we have

$$\frac{\mathrm{d}}{\mathrm{d}m}\,[m\,(\xi - \tanh\xi)] = \xi - \beta\sinh\xi. \tag{35}$$

The exponent is therefore maximum when $\xi = \beta\,\sinh\xi$, an equation that can be solved numerically to obtain the function $\xi(\beta)$. A simple algorithm is to iterate $\xi = \sinh^{-1}(\xi/\beta) \equiv \log\{(\xi/\beta) + [(\xi/\beta)^2 + 1]^{1/2}\}$, which converges very rapidly unless β is close to 1.

The value of m for which $\xi = \beta\,\sinh\xi$ is, by virtue of equation (33),

$$m_0 = \frac{\mu\xi}{(\xi - \tanh\xi)}\,. \tag{36}$$

The value of the second derivative at the point $m = m_0$ is

$$\frac{d^2}{dm^2}\left[m\left(\xi - \tanh\xi\right)\right] = \frac{\zeta(\beta)}{\mu} \tag{37}$$

where

$$\zeta(\beta) = \frac{(\xi\coth\xi - 1)^3}{\xi}. \tag{38}$$

A few typical values are listed below:

$\beta =$	0.01	0.1	0.5	0.9	0.99	$1-\epsilon$
$\xi =$	7.2840	4.4999	2.1773	0.8034	0.2458	$\sqrt{6\epsilon}$
$m_0/\mu =$	1.1591	1.2856	1.8104	5.8439	50.854	$1/2\epsilon$
$\zeta =$	34.068	9.5363	0.8630	0.0110	3×10^{-5}	$(6\epsilon)^{5/2}/27$.

The quadratic term in the exponent of (34) is $-\zeta\,(m-m_0)^2/\mu$, so that the peak has a width of order $\sqrt{\mu}$, while m_0 is of order μ. (These estimates are no longer valid if $\epsilon = 1-\beta \lesssim \mu^{-2/5}$, which is a very small number. We shall not discuss here these extreme conditions). In the next term of the expansion, the third derivative is of order μ^{-2}, and when multiplied by $(m-m_0)^3 \sim \mu^{3/2}$ it gives a result of order $\mu^{-1/2}$, which can be neglected. We can therefore stop the power expansion at the quadratic term, and we obtain, from equations (27), (31) and (34),

$$\Gamma(t) = \frac{\lambda^2\hbar}{2}\sum J_m^2 \simeq \frac{\lambda^2\hbar\,e^{-2m_0(\xi-\tanh\xi)}}{4\pi\,m_0\,\tanh\xi}\int dm\,e^{-\zeta(m-m_0)^2/\mu}. \tag{39}$$

Because of the sharp peak, the limits of integration can be extended to $\pm\infty$, and the integral is $(\pi\mu/\zeta)^{1/2}$. Note that $\Gamma(t)$ no longer depends on t, provided that $r \ll ct \ll R$.

After some rearrangement, we finally get

$$\Gamma = \tfrac{1}{2}\lambda^2\hbar\,e^{-2\mu\xi}\sqrt{\frac{\tanh\xi}{\pi\mu\xi(\xi-\tanh\xi)}}. \tag{40}$$

This result depends on $\mu = \omega/\Omega$, and on $\xi(\beta)$. The dependence on $\mu = \omega/\Omega$ is mostly due to the factor $e^{-2\mu\xi} = e^{-2E\xi/\hbar\Omega}$, where $E = \hbar\omega$ is the energy difference between the two levels of the detector. This exponent is similar to the one in a Boltzmann distribution at a temperature $kT = \hbar\Omega/2\xi$. The latter can be considered as a kind of Unruh temperature for circular motion, in the present model (scalar field in 1+2 spacetime dimensions). The corresponding wavelength is of the order of $c\xi/\Omega = r\xi/\beta$. This is much less than R (if $R \gg ct$) so that thermal equilibrium may be reached within a time t if the coupling constant λ^2 is large enough.

We have checked that the result equation (40), which is asymptotically valid for large R, can also be obtained (with much less labour!) by considering circular motion in an infinite plane, and using the Wightman function for a free field in that plane. We have also checked that, with our choice for the order of magnitude of physical parameters, our conclusions are independent of the boundary condition $\phi = 0$. They are also valid for the more general condition $a\phi + b\,d\phi/dr = 0$.

Appendix. Approximate formulae for Bessel functions

Although the asymptotic properties of Bessel functions are well documented, readily available formulae are not explicit enough for our problem. We briefly derive here the formulae that were needed in this work. Equations labelled (9.*.*) refer to the tables of Abramowitz and Stegun [13]. We neglect in the latter all corrections of order m^{-1}, since $m > \omega/\Omega$ always is a very large number.

First, we note from (9.5.14) that $z_{m1} \simeq m + 1.85575\, m^{1/3}$. Therefore all zeros of $J(x)$ are larger than m. Asymptotic properties of higher zeros can be obtained as follows. From (9.3.3) and (9.3.19) we have

$$J_m(m\ \sec\eta) = \sqrt{\frac{2}{\pi m \tan\eta}}\ \cos\left[m\,(\tan\eta - \eta) - \pi/4\right] \tag{A1}$$

$$J'_m(m\ \sec\eta) = -\sqrt{\frac{\sin 2\eta}{\pi m}}\ \sin\left[m\,(\tan\eta - \eta) - \pi/4\right]. \tag{A2}$$

The sth zero of $J_m(x)$ occurs for a value of η given by

$$m\,(\tan\eta - \eta) - \pi/4 = s\pi - \pi/2 \tag{A3}$$

and its value is $z_{ms} = m\ \sec\eta$. At each zero, the sine in equation (A2) is ± 1, and therefore

$$z_{ms}\,[J'_m(z_{ms})]^2 = 2\,\frac{\sin\eta}{\pi} \tag{A4}$$

where η is given by equation (A3). The explicit value of η can be obtained by iterating $\eta = \tan^{-1}[\eta + \pi\,(s - 0.25)/m]$, which converges very rapidly.

Consecutive zeros (for given m) satisfy

$$\pi = m\,\Delta(\tan\eta - \eta) = m\ \tan^2\eta\,\Delta\eta. \tag{A5}$$

The justification of the last step is that $\Delta\eta$ is a very small number, since m is large. Therefore consecutive values of z_{ms} satisfy

$$\Delta z_{ms} = m\,\Delta\sec\eta = \frac{\pi}{\sin\eta}. \tag{A6}$$

It follows that for any s, large or small, the denominator of C_{ms} in equation (9) is

$$z_{ms}\,[J'_m(z_{ms})]^2 = \frac{2}{\Delta z_{ms}} \tag{A7}$$

a remarkably simple formula which is crucial in our work.

Although we expected these formulae to be correct only to order m^{-1}, we found that they are excellent approximations even for $J_1(x)$. For example, we obtain $z_{11} = 3.7944$ versus the correct value 3.8317. For larger s the agreement is even better.

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References

[1] Davies P C W 1975 *J. Phys. A: Math. Gen.* **8** 609
[2] Unruh W G 1976 *Phys. Rev.* D **14** 870
[3] Bell J S and Leinaas J M 1983 *Nucl. Phys.* B **212** 131
[4] Bell J S, Hughes R J and Leinaas J M 1985 *Z. Phys.* C **28** 75
[5] Bell J S and Leinaas J M 1987 *Nucl. Phys.* B **284** 488
[6] Rogers J 1988 *Phys. Rev. Lett.* **61** 2113
[7] Letaw J R and Pfautsch J D 1980 *Phys. Rev.* D **22** 1345
[8] Grove P G and Ottewill A C 1983 *J. Phys. A: Math. Gen.* **16** 3905
[9] Casimir H B G 1948 *Proc. Kon. Ned. Akad. Wetenschap.* **51** 793
[10] Spaarnay M J 1958 *Physica* **24** 751
[11] Peres A 1992 *Quantum Chaos—Quantum Measurements* ed P Cvitanović, I Percival and A Wirzba (Dordrecht: Kluwer) 249
[12] Levin O, Peleg Y and Peres A 1992 *J. Phys. A: Math. Gen.* **25** 6471
[13] Abramowitz M and Stegun I A 1968 *Handbook of Mathematical Functions* (New York: Dover)